Capacitance of Coupled Circular Microstrip Disks

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Abstract—The coupling between circular disks placed on a grounded dielectric substrate is studied analytically and numerically. The problem is formulated exactly by applying the Kobayashi potential, which uses discontinuity properties of Weber–Schafheitlin integrals, as an electrostatic problem when the potentials on the disks are specified. Numerical results for charge distribution and gap capacitance are presented. The potential distribution on the disks is calculated numerically to check if it satisfies the specified boundary condition.

I. Introduction

THE ELECTROSTATIC problem of a pair of identical circular disk condensers has claimed the attention of numerous investigators over a long span of time [1]–[4]. Approximate solutions of this problem have been derived by Kirchhoff, who refers to earlier papers by Clausius and Helmholtz and improves the previous crude estimate for capacitance by suggesting a plausible edge correction (also, see papers by Maxwell, Ignatowsky, Polya and Szego, and others [1], [2]). An exact solution of this problem also has been attacked, and we can refer to papers by Love, Nicholson [1], and Nomura [3]. A critical review of the approximate solutions has been given by Hutson [2], using Love's integral equation, and by Leppington and Levine [4], using another integral equation.

Recently, the fringing effects on the capacitance of a circular parallel-plate capacitor filled with dielectric has become an important topic because it has application to microstrip circuits [5], [6]. The problem can be formulated rigorously by applying a dual integral equation [5], [6] or applying the Kobayashi potential [7], the name given by Sneddon [1] to the expression for the potential constructed by using the properties of Weber–Schafheitlin integrals proposed by Kobayashi in 1931. For practical purposes the coupling between printed microstrip circuit components is an important problem [8], [9]. As is pointed out by Sneddon, the problem of determining the electrostatic potential of the field due to two equal coplaner electrified disks is a difficult one, so that the problems have been little studied since Kobayashi [10] showed one of the approaches to those kinds of problems. Recently, Uzunoglu and Katechi [11] studied the more general problem of a coupled microstrip resonator using numerical methods and obtained some numerical results for gap capacitance. Their study concentrates on determining a resonant frequency. It is the purpose of this paper to treat this problem rigorously as an electrostatic potential problem and to obtain numerical results for a wider range of physical quantities such as potential distribution and charge distribution, as well as gap capacitance. We followed Kobayashi's procedure as an analytical method. The obtained numerical information will serve as a confirmation of the newly developed analytical technique as well as for the practical design of microwave integrated circuits. To check the validity of the present treatment we compared the calculated potential on the disks with specified values, and agreement between them is satisfactory for practical purposes.

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II. Statement of the Problem

The geometry of the problem is depicted in Fig. 1. Circular disks of the same size are placed on a grounded dielectric substrate. The thickness and relative dielectric constant of the substrate are h and ϵ_r, respectively. The separation between the centers of the disks is pa, where a is radius of the disk. We will consider the case when potential distributions of the disks are specified as $f_1(r_1, \theta_1)$ and $f_2(r_2, \theta_2)$, where (r_1, θ_1) and (r_2, θ_2) are local cylindrical coordinates whose origins are located at the centers of disk 1 and disk 2, respectively. According to the method of the Kobayashi potential, we can assume a potential function in each region as follows:

$$\Phi_1^{(1)} = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(1)} \cos m\theta_1 + B_{mn}^{(1)} \sin m\theta_1 \right) W_1(\rho_1, z)$$

$$\Phi_1^{(2)} = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(2)} \cos m\theta_2 + B_{mn}^{(2)} \sin m\theta_2 \right) W_1(\rho_2, z) \qquad (z \geqslant h) \quad (1)$$

$$\Phi_2^{(1)} = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(1)} \cos m\theta_1 + B_{mn}^{(1)} \sin m\theta_1 \right) W_2(\rho_1, z)$$

$$\Phi_2^{(2)} = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(2)} \cos m\theta_2 + B_{mn}^{(2)} \sin m\theta_2 \right) W_2(\rho_2, z) \qquad (h \geqslant z \geqslant 0) \quad (2)$$

where $\rho_1 = r_1/a$, $\rho_2 = r_2/a$ and, $W_1(\rho, z)$ and $W_2(\rho, z)$ are defined by

$$W_1(\rho_i, z) = \int_0^\infty \frac{J_m(\rho_i \xi) J_{m+2n+1/2}(\xi)}{\sqrt{\xi}} f_i(\xi) \cdot \exp\left[-\frac{z-h}{a}\xi\right] d\xi$$

$$W_2(\rho_i, z) = \int_0^\infty \frac{J_m(\rho_i \xi) J_{m+2n+1/2}(\xi)}{\sqrt{\xi}} g_i(\xi) \cdot \sinh\left(\frac{z}{a}\xi\right) d\xi \qquad (i=1,2) \quad (3)$$

where $f_i(\xi)$ and $g_i(\xi)$ $(i=1,2)$ are unknown functions which are to be determined so that the potential function defined in (2) satisfies the boundary conditions on the surface of the substrate except the conducting disks. In the above equations, $\Phi_1^{(1)}$ and $\Phi_2^{(1)}$ are potential functions when disk 2 is absent, while the function $\Phi_1^{(2)}$ and $\Phi_2^{(2)}$ are those when disk 1 is absent. Imposing the continuity of potential functions and electric flux on the surface of the dielectric substrate, the potential functions of the problem given in Fig. 1 may be expressed as

$$\Phi_1 = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(1)} \cos m\theta_1 + B_{mn}^{(1)} \sin m\theta_1\right) U_1(\rho_1, z) + \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(2)} \cos m\theta_2 + B_{mn}^{(2)} \sin m\theta_2\right) U_1(\rho_2, z) \qquad (z \geqslant h) \quad (4a)$$

$$\Phi_2 = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(1)} \cos m\theta_1 + B_{mn}^{(1)} \sin m\theta_1\right) U_2(\rho_1, z) + \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(A_{mn}^{(2)} \cos m\theta_2 + B_{mn}^{(2)} \sin m\theta_2\right) U_2(\rho_2, z) \qquad (0 \leqslant z \leqslant h) \quad (4b)$$

$$U_1(\rho_i, z) = \int_0^\infty \frac{J_m(\rho_i z) J_{m+2n+1/2}(\xi)}{\sqrt{\xi}} P(\xi, h) \cdot \exp\left[-\frac{z-h}{a}\xi\right] d\xi$$

$$U_2(\rho_i, z) = \int_0^\infty \frac{J_m(\rho_i \xi) J_{m+2n+1/2}(\xi)}{\sqrt{\xi}} P(\xi, z)\, d\xi$$

$$P(\xi, z) = \frac{\sinh\left(\frac{z}{a}\xi\right)}{\epsilon_r \cosh\left(\frac{h}{a}\xi\right) + \sinh\left(\frac{h}{a}\xi\right)} \qquad (i=1,2). \quad (4c)$$

The expansion coefficients A_{mn} and B_{mn} are to be determined so that Φ_1 and Φ_2 reduce to specified potential distributions $f_1(r_2, \theta_1)$ and $f_2(r_2, \theta_2)$ on the disks. Though Φ_1 and Φ_2 in (3) give a general solution when arbitrary potential distributions are specified on each disk, we will restrict ourselves at this stage to the special case of a constant potential on the disk, since the conducting disk is at an equipotential in practical situations. Since the expressions for Φ_1 and Φ_2 are mixed functions of variables (r_1, θ_1) and (r_2, θ_2), they must be transformed to the function of only (r_1, θ_1) or (r_2, θ_2) when we impose the boundary condition on each disk. This is realized by applying the addition theorem of Bessel functions. Setting $\Phi_{1,2}|_{z=h} = V_1$, $(0 < r_1 < a,\ 0 < \theta_1 < 2\pi)$, $\Phi_{1,2}|_{z=h} = V_2$, $(0 < r_2 < a,\ 0 < \theta_2 < 2\pi)$, and using properties of the Fourier series, we obtain the following relations:

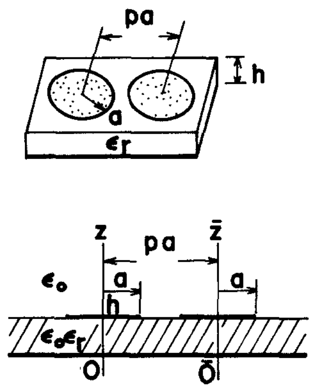

Fig. 1. Geometry of the problem.

$$\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \left[A_{0n}^{(i)} \int_0^\infty \frac{J_0(\rho_i \xi) J_{2n+1/2}(\xi)}{\sqrt{\xi}} P(\xi, h)\, d\xi + \sum_{m=0}^{\infty} A_{mn}^{(j)} \int_0^\infty \frac{J_0(\rho_i \xi) J_m(p\xi) J_{m+2n+1/2}(\xi)}{\sqrt{\xi}} \cdot P(\xi, h)\, d\xi \right] = V_i \quad (5a)$$

$$\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \left[A_{ln}^{(i)} \int_0^\infty \frac{J_l(\rho_i \xi) J_{l+2n+1/2}(\xi)}{\sqrt{\xi}} P(\xi, h)\, d\xi \cdot + \sum_{m=0}^{\infty} A_{mn}^{(j)} \int_0^\infty \frac{J_l(\rho_i \xi) J_{l+2n+1/2}(\xi)}{\sqrt{\xi}} P(\xi, h) \cdot \left\{ J_{l+m}(p\xi) + (-1)^l J_{l-m}(p\xi) \right\} d\xi \right] = 0 \quad (5b)$$

where $j = 2/i$, $(i=1,2)$. Expanding the Bessel functions $J_m(\rho\xi)$ in (5) by Jacobi's polynomials $u_n^m(\rho^2)$ as defined in the Appendix, and using the orthogonality of the polynomials, we derive determinantal equations for expansion coefficients A_{mn}. The results are expressed as

$$\sum_{n=0}^{\infty} A_{0n}^{(i)} GA(0, n, k) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn}^{(j)} H(0, m, n, k) = C_k^{(i)}$$

$$\sum_{n=0}^{\infty} A_{ln}^{(i)} GA(l, n, k) + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn}^{(j)} H(l, m, n, k) = 0$$

$$(l = 1, 2, \cdots, \qquad k = 0, 1, 2, \cdots) \quad (6)$$

where

$$GA(l,n,k)=\int_0^\infty \frac{J_{l+2n+1/2}(\xi)J_{2k+1/2}(\xi)}{\xi}P(\xi,h)\,d\xi$$

$$H(l,n,m,k)=\frac{1}{1+\delta_{0l}}\int_0^\infty \frac{J_{m+2n+1/2}(\xi)J_{l+2k+1/2}(\xi)}{\xi}\cdot P(\xi,h)\{J_{m+l}(p\xi)+(-1)^l J_{m-l}(p\xi)\}\,d\xi$$

$$C_k^{(i)}=V_i\delta_{k0},\qquad j=2/i\qquad (i=1,2). \tag{7}$$

Once the expansion coefficients A_{mn} are determined from (6) the potential at any points is calculated from

$$\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\sqrt{\frac{2}{\pi}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{A_{mn}^{(1)}\begin{pmatrix}U_1(\rho_1,z)\\ U_2(\rho_1,z)\end{pmatrix}\cos m\theta_1+A_{mn}^{(2)}\begin{pmatrix}U_1(\rho_2,z)\\ U_2(\rho_2,z)\end{pmatrix}\cos m\theta_2\right\} \tag{8}$$

where functions $U(\rho,z)$ are defined in (4c). The expressions for charge distribution σ_1 and σ_2 on the surface of disk 1 and disk 2, respectively, are derived from Φ_1 and Φ_2. The expressions for σ_1 and σ_2 are given by

$$\sigma_i(\rho_i,\theta)=-\epsilon_0\left\{\frac{\partial\Phi_1}{\partial z}\bigg|_{z=h}-\epsilon_r\frac{\partial\Phi_2}{\partial z}\bigg|_{z=h}\right\}=\sqrt{\frac{2}{\pi}}\frac{\epsilon_0}{a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^{(i)}\cos m\theta_i\cdot\int_0^\infty\sqrt{\xi}J_m(\rho_i\xi)J_{m+2n+1/2}(\xi)\,d\xi\qquad (i=1,2). \tag{9}$$

The charge density outside of the disk on the surface of the substrate is found to be zero since the above integral is shown to vanish for $\rho_i>1$ using the properties of Weber–Schafheitlin's integral [12]. For $0<\rho_i<1$, the integral can be expressed in terms of a hypergeometric series. The result is

$$\int_0^\infty\sqrt{\xi}J_m(\rho\xi)J_{m+2n+1/2}(\xi)\,d\xi=\frac{\sqrt{2}\,\Gamma(m+n+1)\rho^m}{\Gamma(m+1)\Gamma(n+1/2)}\cdot F\left(m+n+1,\frac{1}{2}-n,m+1,\rho^2\right)=\frac{\sqrt{2}\,\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1/2)}\frac{\rho^m}{\sqrt{1-\rho^2}}F\cdot\left(-n,m+n+\frac{1}{2},m+1,\rho^2\right) \tag{10}$$

where the relation $F(a,b,c;\,z)=(1-z)^{c-a-b}F(c-a,c-b,c;\,z)$ is used to derive the second expression on the right-hand side of the above equation, which is a polynomial of order n. The total charge on the disk is obtained by integrating the charge density over the disk. The total charge of disk 1 is given by

$$Q_1=a^2\int_0^{2\pi}d\theta_1\int_0^1\rho_1\,d\rho_1\sigma_1(\rho_1,\theta_1)=2\sqrt{2\pi}\,a\epsilon_0\sum_{n=0}^{\infty}A_{0n}^{(1)}\int_0^\infty\sqrt{\xi}J_{2n+1/2}(\xi)\,d\xi\cdot\int_0^1 J_0(\rho_1\xi)\rho_1\,d\rho_1=2\sqrt{2\pi}\,a\epsilon_0\sum_{n=0}^{\infty}A_{0n}^{(1)}\int_0^\infty\frac{J_1(\xi)J_{2n+1/2}(\xi)}{\sqrt{\xi}}d\xi=4a\epsilon_0A_{00}^{(1)} \tag{11}$$

where we used the formula [12]

$$\int_0^1\frac{J_1(\xi)J_{2n+1/2}(\xi)}{\sqrt{\xi}}d\xi=\sqrt{\frac{2}{\pi}}\,\delta_{0n}. \tag{12}$$

Similarly, the total charge of disk 2 is expressed as $Q_2=4a\epsilon_0A_{00}^{(2)}$.

III. Numerical Results and Discussion

In this section we present some numerical results for physical quantities. Firstly, we have determined the numerical value of the function GA and H defined in (7) to obtain a solution for A_{mn} from (6). Since it is difficult to calculate the integrals GA and H analytically, we get the results by numerical integration. GA and H are written as

$$GA(n,k)=\int_0^\infty\frac{J_{2n+1/2}(\xi)J_{2k+1/2}(\xi)}{\xi}\cdot\left\{P(\xi,h)-\frac{1}{\epsilon_r+1}\right\}d\xi+GA_0(n,k) \tag{13a}$$

$$H(n,m,k)=\int_0^\infty\frac{J_{2n+m+1/2}(\xi)J_{2k+1/2}(\xi)J_m(\rho\xi)}{\xi}\cdot\left\{P(\xi,h)-\frac{1}{\epsilon_r+1}\right\}d\xi+H_0(n,m,k) \tag{13b}$$

where the integrals $GA_0(n,k)$ and $H_0(n,m,k)$ are performed analytically and are given by

$$GA_0(n,k)=\frac{1}{\epsilon_r+1}\int_0^\infty\frac{J_{2n+1/2}(\xi)J_{2k+1/2}(\xi)}{\xi}d\xi=\frac{1}{\epsilon_r+1}\frac{1}{4n+1}\delta_{nk} \tag{14a}$$

$$H_0(n,m,k)=\frac{1}{\epsilon_r+1}\cdot\int_0^\infty\frac{J_{2n+m+1/2}(\xi)J_{2k+1/2}(\xi)J_m(p\xi)}{\xi}d\xi=\frac{1}{\epsilon_r+1}\frac{(-1)^{n+m}}{4n+2h+1}\sum_{l=0}^{\infty}\left\{\frac{(2l+2n+2k+m+1)!}{l!(l+2n+2k+m+1)!}\cdot\frac{\Gamma(l+n+k+m+\frac{1}{2})\Gamma(l+n+k+\frac{1}{2})}{\Gamma(l+2n+m+\frac{3}{2})\Gamma(l+2k+\frac{3}{2})p^{2l+2n+2k+m}}\right. \tag{14b}$$

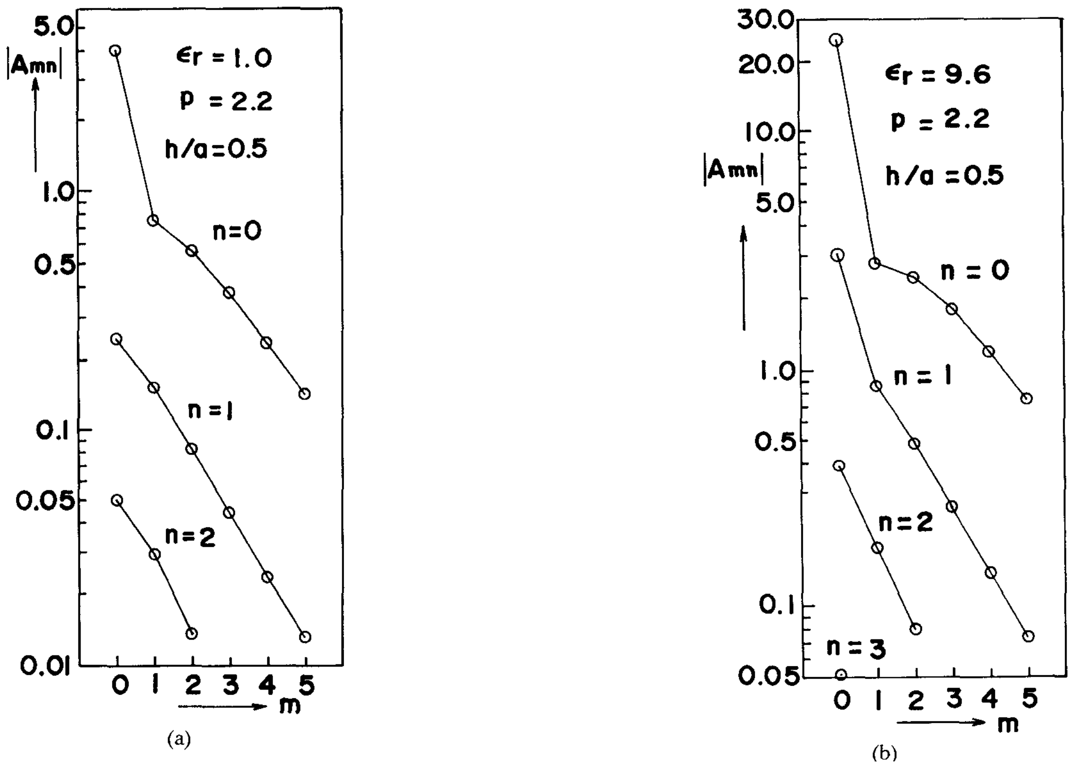

Fig. 2. The variation of expansion coefficients A_{mn} with respect to subscripts m and n. Thickness of the substrate is $h/a = 0.5$ and separation between the disks is $p = 2.2$. (a) $\epsilon_r = 1.0$. (b) $\epsilon_r = 9.6$.

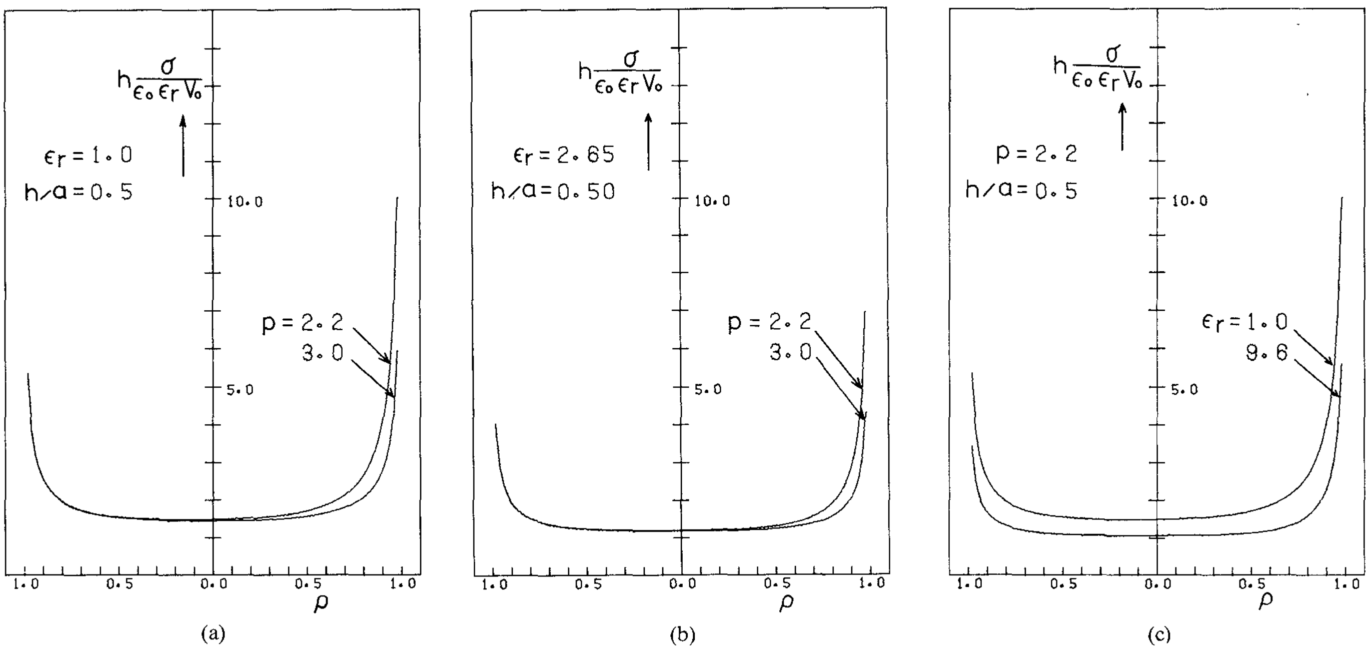

Fig. 3. The charge distribution on disk 1 in the presence of disk 2 when disk 1 and 2 are at potential V_0 and $-V_0$, respectively.

The first integrals in (13) can be truncated by taking a finite range of integration, since the term $P(\xi, h)-1/(1+\epsilon_r)$ decreases exponentially when the value of ξ increases. If the values of GA and H for various values of n, k, and m are obtained, the determination of expansion coefficients A_{mn} is straightforward. A set of equations (6) is solved using a Gauss–Seidel procedure. It is worthwhile noting that the dependency of expansion coefficients A_{mn} on the subscripts m and n, where m and n refer to mode numbers along the circumferential and radial directions, respectively. In Fig. 2, we show the values of $|A_{mn}|$ for various values of m and n when the relative dielectric constant of the substrate is $\epsilon_r = 1.0$ (Fig. 2a) and $\epsilon_r = 9.6$ (Fig. 2b), the thickness of the substrate is $h/a = 0.5$, and the separation between the disks is $p = 2.2$. In each case, the magnitude of A_{mn} decreases more rapidly with n than

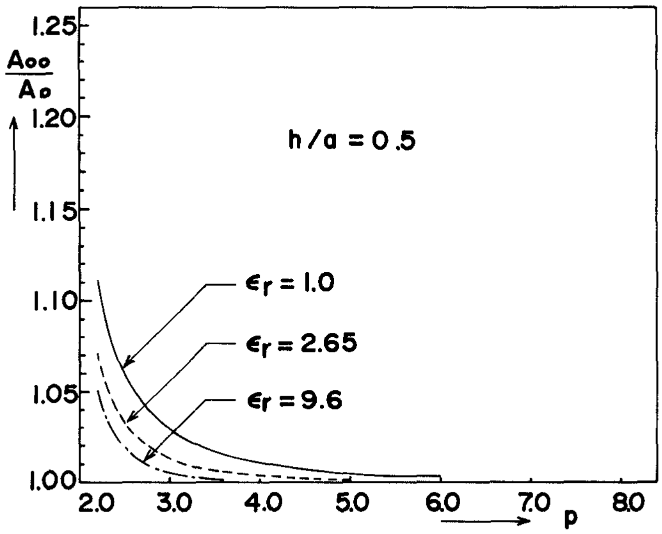

Fig. 4. The rate of increase of total charge distributed on disk 1 electrified to V_0 due to the coupling with disk 2, which is at the potential $-V_0$.

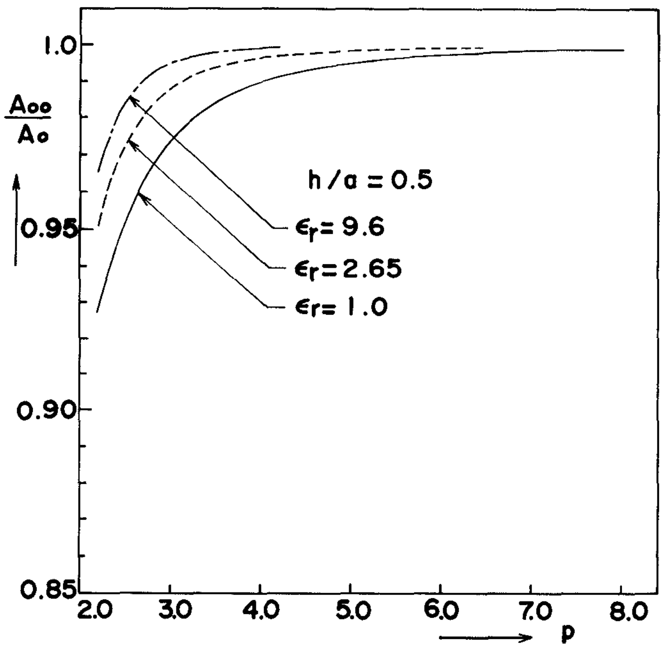

Fig. 5. The rate of decrease of total charge distributed on disk 1 electrified to V_0 due to the coupling with disk 2, which is at the same potential.

with m. Experience shows that the number of modes along the circumferential direction (m) should be roughly four times of that along the radial direction (n) from the convergence point of view, particularly for tight coupling. But the choice was not so critical, and we have not experienced in this problem the phenomenon of relative convergence discussed in [13]. The numerical results for charge distribution on the disks are shown in Fig. 3, when disk 1 is at the potential of V_0 and the disk 2 is at $-V_0$. These figures show the effects of thickness and dielectric constant of the substrate, and of the separation between the disks on the charge distribution on the disk. Each figure depicts the normalized charge distribution along the line connecting the center of each disk. $\rho = 1$ corresponds to the edge of the disk, and the charge density increases near the edge in a manner $(1-\rho^2)^{-1/2}$. When the coupling is rather close, the charge density shows a tendency to concentrate around the edge close to another disk, and the symmetry of charge distribution is broken considerably. The degree of the asymmetry of the charge distribution is one of the measures of the amount of electrostatic coupling between the disks. When the dielectric constant of the substrate is very large or the thickness of the substrate is very small, the coupling effect is little recognized. Since the electric-field flux, which starts from disk 1 and ends on disk 2, increases as the disks come closer, the total charge stored in each

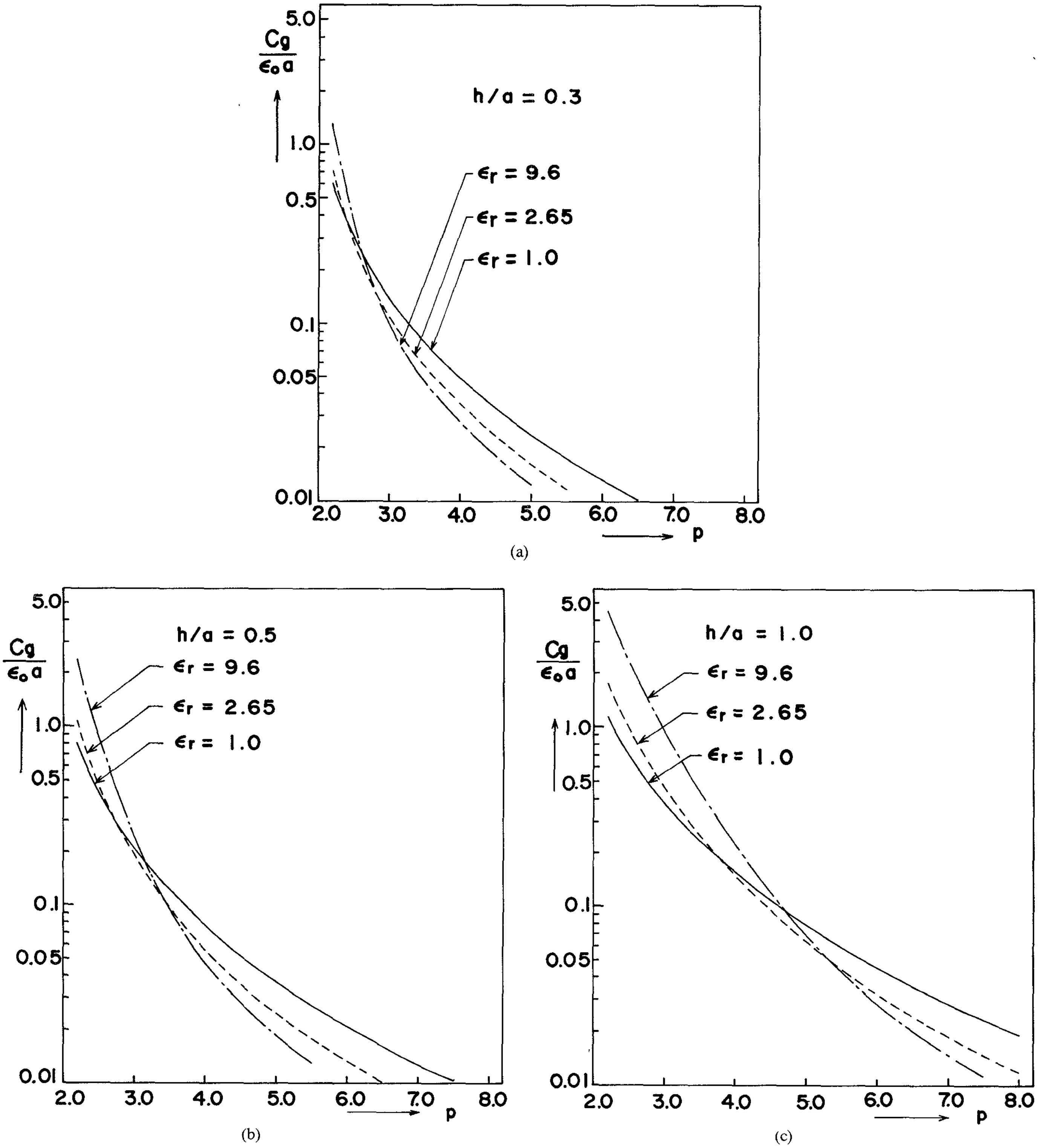

Fig. 6. The variation of gap capacitance C_g with respect to separation between disks p.

disk increases. As shown in (11), the total charge is proportional to the expansion coefficients A_0. The relations between the separation p, relative dielectric constant ϵ_r, and A_{00}/A_0 are shown in Fig. 4, where A_0 is the corresponding expansion coefficient associated with a single disk. On the other hand, when both disks are at the equal potential V_0, the total charge decreases to some limited values as the disks come closer. The behavior is shown in Fig. 5. Uzunoglu and Katechi proposed a gap capacitance C_g as a measure of electrostatic coupling between the disks, which

TABLE I
POTENTIAL DISTRIBUTION ON THE SURFACE OF THE SUBSTRATE ALONG THE LINE JOINING THE CENTERS OF THE DISKS

	POTENTIAL Φ[v] p=2.2 h/a=0.5		
t	ϵ_r=1.0	ϵ_r=2.65	ϵ_r=9.6
1.1	0.000000	0.000000	0.000000
1.0	0.998220	0.998161	0.998113
0.75	0.999581	0.999583	0.999584
0.5	0.999819	0.999743	0.999680
0.25	1.001006	1.001101	1.001177
0	1.001094	1.001300	1.001467
-0.25	1.000209	1.000285	1.000347
-0.5	0.999632	0.999556	0.999495
-0.75	1.000068	1.000076	1.000083
-1.0	0.999551	0.999500	0.999459
-1.25	0.353900	0.306876	0.274874
-1.5	0.195180	0.149021	0.118961
-1.75	0.118062	0.079110	0.055030
-2.0	0.075978	0.044892	0.026747
-2.25	0.051301	0.027033	0.013693
-2.5	0.036010	0.017179	0.007428
-2.75	0.026095	0.011451	0.004292

is defined as follows. Assuming the disks to be at the potential V_0 and $-V_0$, the total charge on each disk will be

$$Q_D = 2C_gV_0 + Q_s \quad \text{or} \quad C_g = \frac{Q_D - Q_s}{2V_0}$$

where $Q_s = CV_0$ is the total charge of single disk when it is raised to potential V_0.

The effects of separation between disks p, relative dielectric constant ϵ_r, and thickness h of the substrate on the gap capacitance C_g are shown in Fig. 6. From these figures, gap capacitances C_g are found to decrease more rapidly with p when the permittivity of the substrate becomes larger. Finally, to check the validity of the present treatment, we calculated the potential distribution along the line joining each center of the disk when the disks are at the potential 1 V and -1 V. The results for $p = 2.2$ and $h/a = 0.5$ are shown in Table I, in which the potential is specified as 1 V in the range $-1 < \rho/a < 1$, as stated above, while the calculated values depart from specified value from 3 or 4 decimals at that range. The agreement improves with increasing p, though it is not shown here for saving space. The accuracy obtained here will be satisfactory for practical calculation.

IV. CONCLUSION

The coupling between the microstrip circular disks placed on a grounded dielectric substrate is studied analytically as an electrostatic potential problem. The problem is formulated using the Kobayashi potential and is reduced to a simultaneous equation for expansion coefficients. Some numerical results for physical quantities such as charge distribution, potential distribution, and gap capacitance are obtained. The obtained results will serve as a basis for practical design of microstrip circuit components or for criterion of newly developed analytical techniques for related problems.

APPENDIX
JACOBI'S POLYNOMIAL $u_n(x)$

Jacobi's polynomial $u_n(x)$ used in this paper is defined by

$$u_n^m(x) = \sqrt{\frac{2}{\pi}}\, x^{-m/2} \int_0^\infty \frac{J_m(\sqrt{x}\,\xi)\, J_{m+2n+1/2}(\xi)}{\sqrt{\xi}}\, d\xi$$
$$= \frac{\Gamma(n+m+1/2)}{\sqrt{\pi}\,\Gamma(n+1)\Gamma(n+m+1)} x^{-m}(1-x)^{1/2} \frac{d^n}{dx^n} \cdot \left\{x^{n+m}(1-x)^{n-1/2}\right\}.$$

The bessel function $J_m(p\sqrt{x})$ is expanded by the polynomial $u_n^m(x)$ as

$$x^{-m/2} J_m(p\sqrt{x}) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (2m+4n+1) \cdot \frac{J_{m+2n+1/2}(p)}{\sqrt{p}} u_n^m(x).$$

The orthogonality relation for $u_n^m(x)$ is given by

$$\left[\int_0^1 x^m (1-x)^{-1/2} u_n^m(x) u_{n'}^m(x)\, dx = \frac{2\Gamma(n+\frac{1}{2})\Gamma(n+m+\frac{1}{2})}{\pi(2m+4n+1)\Gamma(n+1)\Gamma(n+m+1)} \delta_{nm'}.\right]$$

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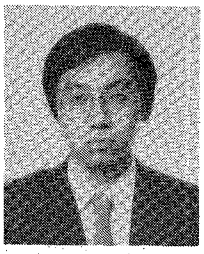

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Quasistatic Characteristics of Covered Coupled Microstrips on Anisotropic Substrates: Spectral and Variational Analysis

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Abstract —**In this paper, expressions to compute the upper and lower bounds on true values of the even- and odd-mode capacitances of covered coupled microstrips over anisotropic substrates are obtained by using the Fourier transform and the variational approach. The method provides accurate calculation and yields the margins of error in the computation. Some examples are shown.**

I. INTRODUCTION

IN RECENT YEARS, the boundary value problems involving microstrip lines on anisotropic substrates have been approached from numerical [1] and analytical points of view [2]–[9]. Alexopoulos *et al.* [2], [3] have shown the effect of an anisotropic substrate on the characteristics of covered coupled microstrips by using the method of moments. Methods for calculating the parameters of single [4], [5] and coupled microstrip lines [6]–[8] have been performed by applying transformation from anisotropic to isotropic problems. Green's functions for examples with anisotropic medium have been obtained using the image-coefficient method in [9].

The spectral-domain approach has been used extensively on problems of microstrip lines on isotropic substrates, and variational expressions of capacitances have been reported [10], [11]. This method was extended by [12] and [13] to analyze the characteristic parameters of single and coupled microstrips on anisotropic substrates.

The purpose of this paper is to solve the variational problem involving covered coupled microstrips on anisotropic substrates with an arbitrary permittivity tensor by using the Fourier transform, and obtaining in this way stationary expressions to compute the upper and lower bounds of the mode quasi-static characteristics of this structure. The method shows the equivalence between the mode capacitances of this structure and another with an isotropic substrate, in agreement with the reported results [14]. Besides, it is a fast and accurate calculation method in most practical cases and it yields the margins of error in the computation.

II. ANALYSIS

Consider the configuration of covered coupled microstrips shown in Fig. 1, which comprises two zero-thickness strips on an anisotropic dielectric substrate, which permit-

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